MATH 1321
MIDTERM EXAM

BY Anan Elayan

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Student Number:
Discussion Instructor:
Discussion Section:

Question 1 (3 points each) Circle the most correct answer:

1. The sequence whose nth term is $a_n = \sqrt[n]{4^n n}$

   (a) diverges
   (b) converges to 0
   (c) converges to 4
   (d) converges to 2

2. The series $\sum_{n=1}^{\infty}(1 - \frac{1}{2n})^n$

   (a) converges by the nth term test
   (b) diverges by the nth term test
   (c) diverges by the root test
   (d) converges by the root test

3. The sequence whose nth term is $a_n = 1 - \cos(\frac{1}{n})$

   (a) diverges
   (b) converges to $1 - \frac{\pi}{2}$
   (c) converges to 1
   (d) converges to 0

4. The series $\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^{\frac{5}{2}}}$

   (a) diverges the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$
   (b) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$
   (c) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
   (d) converges by the nth term test

5. If we use S_4 to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ then the error satisfies

(a) the error is positive and $|\text{error}| < 0.1$

(b) the error is negative and $|\text{error}| < 0.25$

(c) the error is negative and $|\text{error}| < 0.2$

(d) the error is positive and $|\text{error}| < 0.2$

6. The integral $\int_2^{\infty} \frac{2}{x^3 - x}\, dx$

(a) converges by the direct comparison test with $\int_2^{\infty} \frac{1}{x^3}\, dx$

(b) converges by the limit comparison test with $\int_2^{\infty} \frac{1}{x}\, dx$

(c) converges by the limit comparison test with $\int_2^{\infty} \frac{1}{x^3}\, dx$

(d) diverges by the limit comparison test with $\int_2^{\infty} \frac{1}{x^3}\, dx$

7. The sequence whose nth term is $a_n = \frac{n}{\ln n}$

(a) diverges

(b) converges to 0

(c) converges to 1

(d) converges to 2

8. The series $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2 + 1}{n^2 - 5}$

(a) converges absolutely

(b) converges conditionally

(c) diverges by the nth term test

(d) converges by the nth term test

9. The series $\sum_{n=1}^{\infty}(-1)^n \frac{5}{3^n}$

(a) diverges
(b) converges to $-\frac{5}{4}$
(c) converges to $\frac{15}{4}$
(d) converges to $-\frac{5}{2}$

10. The series $\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$

(a) diverges
(b) converges to $\frac{7}{2}$
(c) converges to $\frac{1}{2}$
(d) converges to $-\frac{1}{2}$

11. The integral $\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

(a) converges by the direct comparison test with $\int_2^{\infty} \frac{dx}{x}$
(b) converges by the limit comparison test with $\int_2^{\infty} \frac{dx}{x^2}$
(c) diverges by the direct comparison test with $\int_2^{\infty} \frac{dx}{x}$
(d) diverges by the limit comparison test with $\int_2^{\infty} \frac{dx}{x^2}$

12. The series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

(a) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$
(b) diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(c) diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$
(d) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

13. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(a) diverges by the integral test
(b) converges by the integral test
(c) diverges by the nth term test
(d) converges by the nth term test

14. The series $\sum_{n=1}^{\infty} \frac{\frac{1}{2}\tan^{-1} n}{n^3+1}$

(a) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$
(b) diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$
(c) diverges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$
(d) converges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

15. The sequence whose nth term is $a_n = n\tan^{-1} n$

(a) diverges
(b) converges to 0
(c) converges to $\frac{\pi}{2}$
(d) converges to $-\frac{\pi}{2}$

16. The series $\sum_{n=2}^{\infty} \frac{(\ln n)^{35}}{n!}$

(a) diverges by the ratio test
(b) converges by the ratio test
(c) diverges by the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n!}$
(d) diverges by the nth term test

17. The series $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

(a) converges by the root test

(b) diverges by the root test

(c) converges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{5^n}$

(d) diverges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{5^n}$

18. One of the following is true

(a) If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges

(b) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges

(c) If $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

(d) $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge

19. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(a) converges absolutely

(b) converges conditionally

(c) diverges by the nth term test

(d) converges by the nth term test

20. The series $\sum_{n=0}^{\infty} e^{-n}$

(a) diverges

(b) converges to $\frac{e}{e-1}$

(c) converges to $\frac{1}{1-e}$

(d) converges to $\frac{e-1}{e}$

21. If $\sum a_n$ is a convergent series of positive terms, then the series $\sum (a_n)^n$ converges

(a) True

(b) False

22. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+19}$. The least number of terms that are needed to estimate the sum of the series with an error of less than 0.01 is

(a) five terms

(b) fifteen terms

(c) nine terms

(d) ten terms

23. The series $\sum_{n=1}^{\infty}(x-1)^n$

(a) converges conditionally for $0 < x < 2$

(b) converges absolutely for $0 \leq x \leq 2$

(c) converges absolutely for $0 < x < 2$

(d) converges conditionally for $0 \leq x \leq 2$

24. The integral $\int_1^2 \frac{dx}{(x-1)^{\frac{3}{2}}}$

(a) diverges

(b) converges to 0

(c) converges to 1

(d) converges to $-\frac{1}{2}$

Question 2 (10 points) Evaluate the integral $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

$$= \int_0^1 + \int_1^2$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{\frac{1}{3}}\Big]_0^b$$

$$= \lim_{b\to 1^-} \left(3\sqrt[3]{b-1} - 3\sqrt[3]{-1}\right) = 3$$

$$\int_1^2 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^2 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} 3(x-1)^{\frac{1}{3}}\Big]_a^2$$

$$= \lim_{a\to 1^+} \left(3\sqrt[3]{2-1} - 3\sqrt[3]{a-1}\right) = 3$$

$$\Rightarrow \int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}} = 3+3 = 6$$

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Question 3 (14 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,2^n}$. Answer the following questions:

1. For what values of x does the series converge absolutely?
2. Find the radius of convergence.
3. For what values of x does the series converge conditionally?
4. Find the interval of convergence.

Apply the ratio test to $\sum |\ \ |$

$$\frac{|x-3|^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{|x-3|^n} = \frac{n}{n+1}\,\frac{|x-3|}{2} \xrightarrow{n\to\infty} \frac{|x-3|}{2}$$

the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,2^n}$ Conv. abs. if $\frac{|x-3|}{2} < 1$

$\iff |x-3| < 2$

$\iff 1 < x < 5$

$x=1 \longrightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, Conv. Cond. by the alt. series test.

$x=5 \longrightarrow \sum_{n=1}^{\infty} \frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, div. (harmonic series)

1. $1 < x < 5$
2. $R = 2$
3. $x = 1$
4. $1 \le x < 5$

Question 4 (10 points) Find the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 2$. (Write the final answer using the sigma notation).

$f(x) = x^{-2} \longrightarrow f(2) = \frac{1}{2^2}$

$f'(x) = -2x^{-3} \longrightarrow f'(2) = \frac{-2}{2^3}$

$f''(x) = (-2)(-3)x^{-4} \longrightarrow f''(2) = \frac{(-2)(-3)}{2^4}$

$\vdots \qquad \vdots$

$f^{(k)}(2) = (-1)^k \frac{(k+1)!}{2^{k+2}}$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)(x-2)^k}{2^{k+2}}$$

